Module 1

Introduction

OR is a scientific approach for decision making. It is a set of operations to be performed to get the desired outcome. We are finding the best operations giving the best outcome.

Ex: 1) In case of industry marketing strategies for product (Research of already existing products, survey, advertising plans etc.)
2) For selecting investment plan with maximum interest.

Phases of Operations Research

1) Formulation : Formulating the problem in an appropriate model (problem statement). This includes finding objective function, constraints or restrictions, alternate course of action, controllable and uncontrollable variables. Right formulation gives right solution. Therefore we must be very careful while executing this phase.

2) Phase 2 - Constructing the mathematical model
This phase is concerned with reformation of the problem in an appropriate form. Mathematical model should include decision variables, objective functions and constraints.

3) Phase 3 - Derivation of the solution from model

This phase involves computation of the value of decision variables which maximise or minimize the objective function. We need to find the optimal solution of the problem.

4) <u>Testing the mathematical model and its solution</u>

The completed model is tested for errors. A good model should be applicable for a longer time and it should be updated time to time by taking into account the past, present and future specifications of the problem.

5) <u>Establishing control over the solution</u>

After testing the next step is to install the well documented system for applying the model. It includes the solution procedure and operating procedure for implementation. This phase also establishes a systematic procedure for detecting changes and controlling the situation.

6) <u>Implemntation</u>

It involves translation of the model's results into operating instructions. It is important to ensure that the solution is accurately translated into an operating procedure to rectify faults in the solution.

Mathematical formulation of a LPP.

Step 1: Define the decision variable $x_1, x_2 \ldots x_n$

Step 2: Construct the objective function which has to be optimized as a linear equation involving decision variables. (maximization or minimization)

Step 3: Express every condition as a linear inequality involving decision variables (constraints)

Step 4: State the nonnegativity condition and hence express the given problem as a mathematical model.

Ex: (1) A firm has 3 products P1, P2, P3 which are produced on 3 different machines. Following table gives the data about products and machine relation. Profit per each unit of product P1, P2 and P3 is Rs. 4, Rs. 3 and Rs. 6 respectively. Formulate this as a mathematical model to find the no. of units of each type to be produced and to maximize the profit.

Machine	Time/unit (minutes)			machine capacity (in min)
	P1	P2	P3	
M1	2	3	2	440
M2	4	—	3	470
M3	2	5	—	430

→ Profit – P1 → 4 Rs.
P2 → Rs. 3
P3 → Rs. 6 } per unit

1) Let x_1 units of P1 produced
x_2 units of P2 produced
x_3 units of P3 produced

2) Objective function → profit → maximize
maximize $Z = 4x_1 + 3x_2 + 6x_3$

3) Constraints
$2x_1 + 3x_2 + 2x_3 \leq 440$
$4x_1 + 3x_3 \leq 470$
$2x_1 + 5x_2 \leq 430$

4) Non negativity
$x_1, x_2, x_3 \geq 0$

(2) The manufacturer produces 3 models M1, M2 and M3 of a certain product using raw materials A and B. The following table gives the data

Raw material	Requirement/unit M1	M2	M3	machine capacity (in minutes)
A	2	3	5	4000
B	4	2	7	6000
min Demand	200	200	150	—
Profit/unit	30	20	50	—

→ Profit – M1 → 30
M2 → 20

M3 → 50

1) Let x_1 units of M1 produced
x_2 units of M2 produced
x_3 units of M3 produced

2) objective function → profit → maximize
max $Z = 30x_1 + 20x_2 + 50x_3$

3) Constraints
$2x_1 + 3x_2 + 5x_3 \leq 4000$
$4x_1 + 2x_2 + 7x_3 \leq 6000$
$x_1 \geq 200$ (minimum demand for x_1)
$x_2 \geq 200$
$x_3 \geq 150$

4) Non negativity
$x_1, x_2, x_3 \geq 0$

③ Manufacture of patent medicines has planned to prepare a production plan for medicine A and B. There are sufficient ingradients available to make 20,000 bottles of medicine A and 40,000 bottles of medicine B. But there are only 45,000 bottles into which the medicines can be filled. Further it takes 3 hours to prepare enough material to fill 1000 bottles of medicine A and 1 hour to prepare material to fill 1000 bottles of medicine B and there are 66 hours available for this operation. The profit is Rs. 8 per bottle of medicine A and Rs.7 per bottle of medicine B. Formulate this as a LPP

→ profit – A → Rs. 8

B → Rs. 7

1) Let x_1 1000 bottles of medicine A produced

x_2 1000 bottles of medicine B produced

2) objective function → profit → maximize

max $Z = 8000x_1 + 7000x_2$

3) Constraints

$x_1 + x_2 \leq 45$

$x_1 \leq 20$

$x_2 \leq 40$

$3x_1 + x_2 \leq 66$

4) Non negativity

$x_1, x_2 \geq 0$

(4) A farmer has 100 Acre farm. He can sell all tomatos, lettuce, radishes which he grow. The price he can obtain is Rs.1 per kg tomato, Rs. 0.75 a head for lettuce, Rs. 2 per kg for radish. The average yield for acre is 2000 kg of tomato, 3000 heads of lettuce and 1000 kg of radishes. Fertilizer is available at Rs. 0.5 per kg and amount required ~~is~~ per acre is 100 kg each for tomato and lettuce, 50 kg for radishes, labour required is 5 man days

for tomatos and radishes, 6 man days for lettuce. A total of 400 man days of labour are available at Rs. 20 per man day. Formulate this as a ~~linear~~ LPP to maximize the profit

→ Let x_1 – acre of tomato
x_2 – acre of lettuce
x_3 – acre of raddish

total sale profit $= 2000 \times 1 \times x_1 + 3000 \times 0.75 \times x_2 + 1000 \times 2 \times x_3$

Fertilizer cost $= 100 x_1 \times 0.5 + 0.5 \times 100 \times x_2 + 0.5 \times 50 \times x_3$

Labour cost $= 5 \times 20 \times x_1 + 6 \times 20 \times x_2 + 5 \times 20 \times x_3$

final profit = sale profit − (fertilizer cost + labour cost)

$$Z = 2000 x_1 + 2250 x_2 + 2000 x_3 - (100 x_1 + 120 x_2 + 100 x_3 + 50 x_1 + 50 x_2 + 25 x_3)$$

$$= 2000 x_1 + 2250 x_2 + 2000 x_3 - (150 x_1 + 170 x_2 + 105 x_3)$$

$$= 1850 x_1 + 2080 x_2 + 1875 x_3$$

Constraints

$$x_1 + x_2 + x_3 \leq 100$$
$$5x_1 + 6x_2 + 5x_3 \leq 400$$
$$x_1, x_2, x_3 \geq 0$$

5) A toy company manufactures 2 types of dolls, a basic version doll A and delux version doll B. Each doll of type B takes twice as long to produce as one of type A, and the company has

time to make a maximum of 2000 per day. The supply of plastic is sufficient to produce 1500 dolls per day. The delux version requires a fancy dress of which there are only 600 per day are available. If the company makes a profit of Rs. 3 and Rs. 5 per doll respectively on doll A and doll B. Then how many of each doll should be produced per day inorder to maximize the profits.

→ Let x_1 - no. of type doll A

x_2 - no. of type doll B

objective function

$$\max z = 3x_1 + 5x_2$$

constraints

$$x_1 + x_2 \le 1500$$

$$x_2 \le 600$$

time required for A $= t_1$

" " B $= 2t_1$

$$t_1 x_1 + 2t_1 x_2 \le 2000$$

ignore $t_1 \Rightarrow x_1 + 2x_2 \le 2000$

$$x_1, x_2 \ge 0$$

Graphical Method

Ex: 1) Obtain graphical solution for the following problem.

maximize $Z = 3x_1 + 5x_2$

s.t (s.t - subjected to the constraints)

$$3x_1 + 2x_2 \le 18$$

$$x_1 \le 4$$

(*)

$2x_2 \le 12$

$x_1, x_2 \geq 0$

$\rightarrow$ $3x_1 + 2x_2 \le 18$

change the inequality to equality

$3x_1 + 2x_2 = 18$

solve for x_1 and x_2 by alternating considering other variable zero

Put $x_1 = 0$

$2x_2 = 18 \Rightarrow x_2 = 9$ (0, 9)

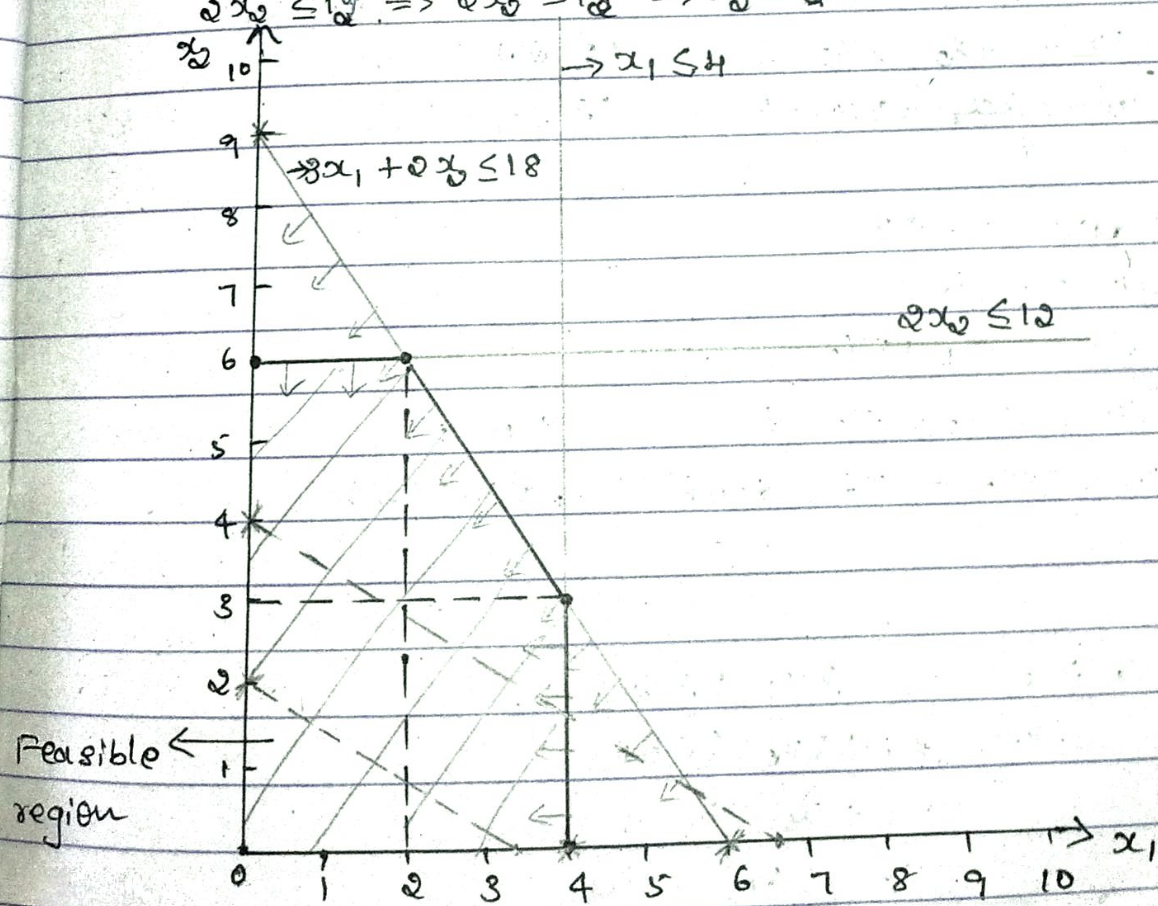
Put $x_2 = 0$

$3x_1 = 18 \Rightarrow x_1 = 6$ (6, 0)

graph

$x_1 \le 4 \Rightarrow x_1 = 4$

$2x_2 \le 12 \Rightarrow 2x_2 = 12 \Rightarrow x_2 = 6$



2 methods

1) Corner point method (CPF)

Note down corner points in feasible region

$(0,0) \Rightarrow z = 0$

$(0,6) \Rightarrow z = 12$

$(4,0) \Rightarrow z = 30$

$(2,6) \Rightarrow z = 36$

$(4,3) \Rightarrow z = 27$

36 is maximum i.e $z = 36 \Rightarrow x_1 = 2$, $x_2 = 6$.

2) Parallel line techniques

take $z = 10$

take $x_1 = 0$, $x_2 = ?$

$x_2 = 0$, $x_1 = ?$

$10 = 3x_1 + 5x_2$

$10 = 5x_2 \Rightarrow x_2 = 2$ $(0, 2)$

$10 = 3x_1 \Rightarrow x_1 = 3.33$ $(3.33, 0)$

take $z = 20$

put $x_1 = 0$

$20 = 3x_1 + 5x_2$

$20 = 5x_2 \Rightarrow x_2 = 4$ $(0, 4)$

put $x_2 = 0$, $20 = 3x_2 \Rightarrow x_2 = 6.66$ $(6.66, 0)$

2) maximize $z = 3x_1 + 5x_2$

s.t $x_1 + 2x_2 \leq 2000$

$x_1 + x_2 \leq 1500$

$x_2 \leq 600$

$x_1, x_2 \geq 0$

$\rightarrow x_1 + 2x_2 \le 2000$

$x_1 + 2x_2 = 2000$

Put $x_1 = 0$

$2x_2 = 2000 \Rightarrow x_2 = 1000 \quad (0, 1000)$

Put $x_2 = 0$

$x_1 = 2000 \quad (2000, 0)$

$x_1 + x_2 \le 1500 \Rightarrow x_1 + x_2 = 1500$

Put $x_1 = 0$

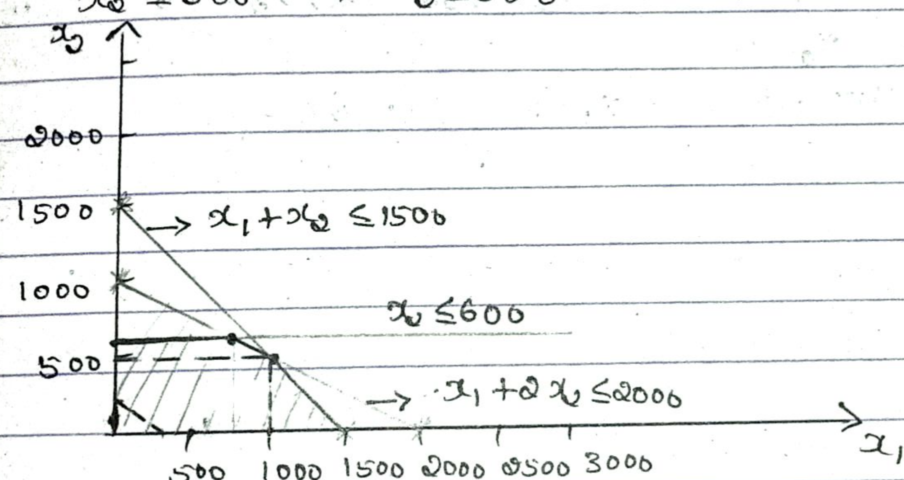
$x_2 = 1500 \quad (0, 1500)$

$x_2 = 0$

$x_1 = 1500 \quad (1500, 0)$

graph

$x_2 \le 600 \Rightarrow x_2 = 600$



Parallel line technique

Let $Z = 1000$

$1000 = 3x_1 + 5x_2$

$x_1 = 0 \Rightarrow x_2 = 200 \quad (0, 200)$

$x_2 = 0 \Rightarrow x_1 = 333.33 \quad (333.33, 0)$

$x_1 = 1000 \quad x_2 = 500$

$Z = 3x_1 + 5x_2 \Rightarrow Z = 5500$

4) A company produces both exterior & interior paints from raw material m_1 and m_2. The following table gives the basic data using the details. Find the optimal solution graphically.

Raw material	Tons of raw material/ton of		max delay availability
	Exterior paint	Interior paint	
m_1	6	4	24
m_2	1	2	6
profit / ton	5	4	

→ Market survey indicates that the daily demand for interior paint cannot exceed for exterior paint by more than one ton. Also the maximum daily demand for interior paints is 2 tons.

→ let x_1 - ton of ~~type~~ exterior paint ~~tf~~

x_2 - ton of interior paint

objective function max $Z = 5x_1 + 4x_2$

5)

Constraints

$$6x_1 + 4x_2 \le 24$$
$$x_1 + 2x_2 \le 6$$
$$x_2 - x_1 \le 1$$
$$x_2 \le 2$$
$$x_1, x_2 \ge 0$$

$$6x_1 + 4x_2 \le 24 \Rightarrow 6x_1 + 4x_2 = 24$$

Put $x_1 = 0$

$4x_2 = 24 \Rightarrow x_2 = 6$ $(0, 6)$

Put $x_2 = 0$

$6x_1 = 24 \Rightarrow x_1 = 4$ $(4, 0)$

$$x_1 + 2x_2 \le 6 \Rightarrow x_1 + 2x_2 = 6$$

Put $x_1 = 0$

$2x_2 = 6 \Rightarrow x_2 = 3$ $(0, 3)$

$x_2 = 0$

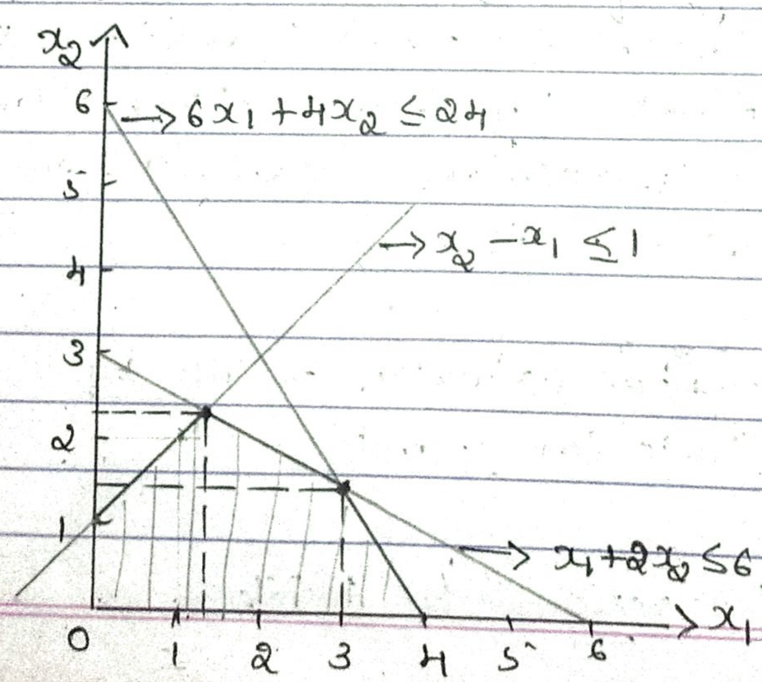
$x_1 = 6$ $(6, 0)$

$$x_2 - x_1 \le 1 \Rightarrow x_2 - x_1 = 1$$

Put $x_1 = 0 \Rightarrow x_2 = 1$ $(0, 1)$

$x_2 = 0 \Rightarrow x_1 = -1$

$x_2 = 2 \Rightarrow x_1 = 1$ $(1, 2)$ (ignore above -ve sign)



Corner point method

$(0,0) \Rightarrow z = 0$

$(0,1) \Rightarrow z = 4$

$(1.33, 2.3) \Rightarrow z = 15.7$

$(3, 1.5) \Rightarrow z = 21$

$(4,0) \Rightarrow z = 20$

21 is maximum i.e $z = 21 \Rightarrow x_1 = 3 \quad x_2 = 1.5$

(*) 5] Old hens can be bought at Rs. 2 each and younger one at Rs. 5 each. The old hens lay 3 eggs per week whereas younger ones lay 5 eggs per week. Each egg being worth 30 paise. The hen costs Rs. 1 per week to feed. If there is only 80 rupees available for purchasing the hens, how many of each kind you will buy to get a profit of more than Rs. 6 per week assuming that you cannot have more than 20 hens. Formulate this and solve graphically.

→ Let $x_1 \rightarrow$ no. of old hens

$x_2 \rightarrow$ no. of younger hens.

objective ~~function~~ $\Rightarrow$ gain = by selling eggs

~~max~~ $= (3x_1 \times 5x_2) \times 0.30$

$\text{cost} = (x_1 + x_2) \times 1$

$\max z = (3x_1 + 5x_2) \times 0.30 - (x_1 + x_2)$

$z = 0.5x_2 - 0.1x_1$

s.t. $x_1 + x_2 \le 20$

$2x_1 + 5x_2 \le 80$

$x_1, x_2 \ge 0$

$x_1 + x_2 = 20$

Put $x_1 = 0 \Rightarrow x_2 = 20$ $(0, 20)$

Put $x_2 = 0 \Rightarrow x_1 = 20$ $(20, 0)$

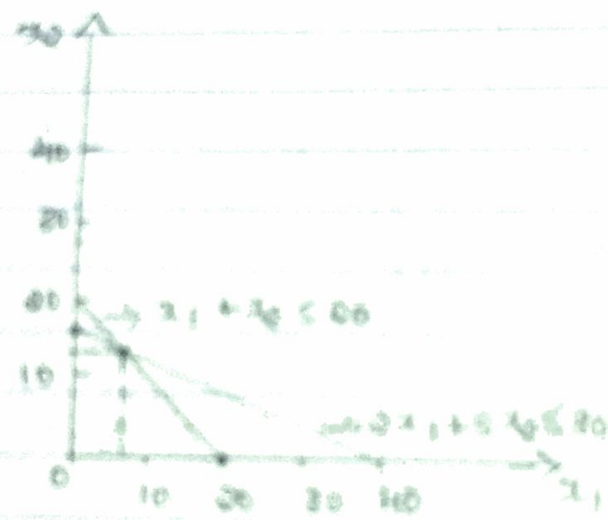
$2x_1 + 5x_2 \le 80$

Put $x_1 = 0$

$5x_2 = 80 \Rightarrow x_2 = 16$ $(0, 16)$

Put $x_2 = 0$

$2x_1 = 80 \Rightarrow x_1 = 40$ $(40, 0)$



Corner point method

$(0,0) \Rightarrow Z = 0$

$(0,16) \Rightarrow Z = 8$

$(6,13) \Rightarrow Z = 5.9$

$(20,0) \Rightarrow Z = 2$

8 is maximum i.e. $Z = 8 \Rightarrow x_1 = 0$ $x_2 = 16$

7) A firm plans to purchase atleast 200 quintal of scrap containing high quality metal X, low quality metal Y. It decides that a scrap purchased must contain atleast 100 quintal of metal X & not more than 35 quintal of metal Y. The firm can purchase the scrap from 2 suppliers A and B. The percentage of X and Y metal by suppliers A & B is given in the following table.

~~Sup~~metals	supplier A	supplier B
X	25%	75%
Y	10%	20%

The price of A's is Rs. 200 per quintal and that of B's is Rs. 400 per quintal. Formulate this & solve graphically to minimize the purchase cost. How much amount of metal should be purchased from supplier A & B?

→ let x_1 - amount of high ~~quality~~ ~~metal~~ X

x_2 " low " Y

let x_1 quintal from supplier A

x_2 " " " B.

$$\max Z = 200x_1 + 400x_2$$

s.t $x_1 + x_2 \geq 200$

$$\frac{25}{100}x_1 + \frac{75}{100}x_2 \geq 100 \Rightarrow \frac{1}{4}x_1 + \frac{3}{4}x_2 \geq 100 \times 4$$

$$x_1 + 3x_2 \geq 400$$

$$\frac{10}{100}x_1 + \frac{20}{100}x_2 \geq 35$$

$$\Rightarrow 0.1x_1 + 0.2x_2 \leq 35$$

$x_1 + 2x_2 \le 350$

$x_1 + 3x_2 = 400$

Put $x_1 = 0$

$3x_2 = 400 \Rightarrow x_2 = 133.33$ $(0, 133.33)$

$x_2 = 0 \Rightarrow x_1 = 400$ $(400, 0)$

$x_1 + x_2 = 200$

Put $x_1 = 0 \Rightarrow x_2 = 200$ $(0, 200)$

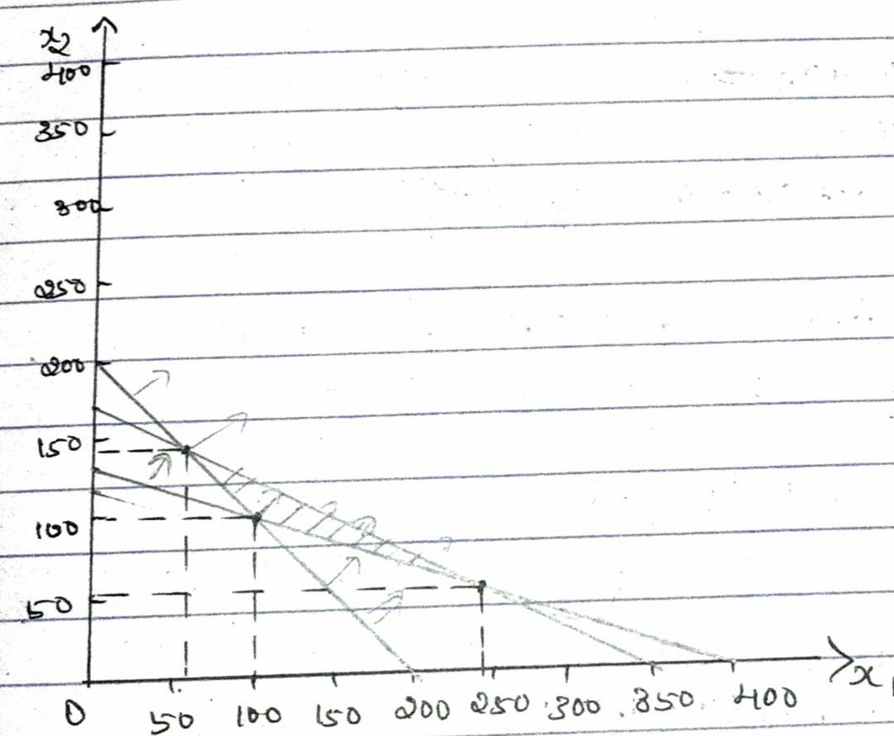
$x_2 = 0 \Rightarrow x_1 = 200$ $(200, 0)$

$x_1 + 2x_2 = 350$

Put $x_1 = 0$

$2x_2 = 350 \Rightarrow x_2 = 175$ $(0, 175)$

$x_2 = 0 \Rightarrow x_1 = 350$ $(350, 0)$



$(100, 100) \Rightarrow Z = 60{,}000$

$(60, 140) \Rightarrow Z = 68{,}000$

$(240, 60) \Rightarrow Z = 72{,}000$

60,000 is minimum i.e $Z = 60{,}000$, $x = 100$, $y = 100$

8) minimize $Z = 0.4x_1 + 0.5x_2$

s.t $0.3x_1 + 0.1x_2 \le 2.7$

$0.5x_1 + 0.5x_2 = 6$

$0.6x_1 + 0.4x_2 \ge 6$

$\rightarrow$ $0.3x_1 + 0.1x_2 = 2.7$

Put $x_1 = 0$ $\quad 0.1x_2 = 2.7 \Rightarrow x_2 = 27$ $\quad (0, 27)$

$x_2 = 0$ $\quad 0.3x_1 = 2.7 \Rightarrow x_1 = 9$ $\quad (9, 0)$

$0.5x_1 + 0.5x_2 = 6$

Put $x_1 = 0$

$0.5x_2 = 6 \Rightarrow x_2 = 12$ $\quad (12, 0)$

$x_2 = 0$

$0.5x_1 = 6 \Rightarrow x_1 = 12$ $\quad (0, 12)$

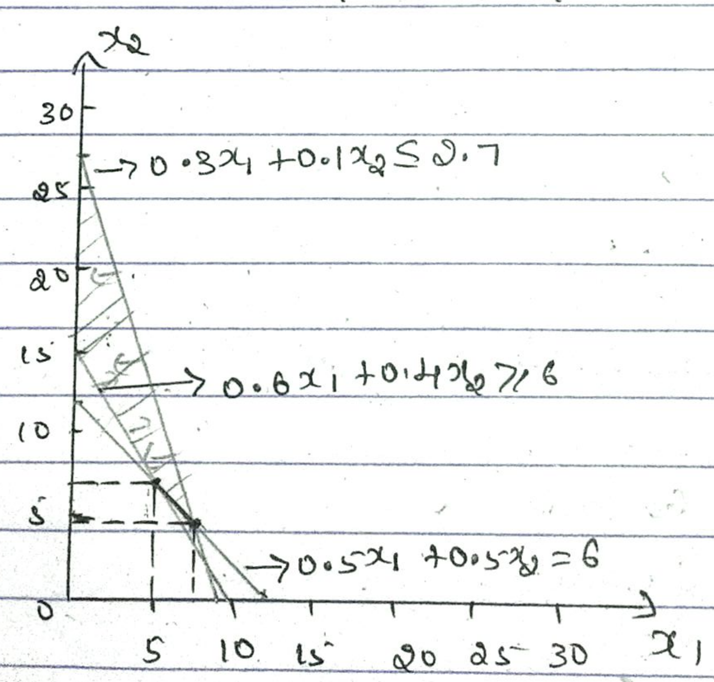
$0.6x_1 + 0.4x_2 \ge 6$

Put $x_1 = 0$

$0.4x_2 = 6 \Rightarrow x_2 = 15$ $\quad (0, 15)$

$x_2 = 0$

$0.6x_1 = 6 \Rightarrow x_1 = 10$ $\quad (10, 0)$



(0, 27)
(9, 0)

(0, 15)
$(5, 7) \Rightarrow z = 5.\cancel{45}5$
$(7.5, 5) \Rightarrow z = 5.5$

$z = 5.5 \quad x_1 = 7.5 \quad x_2 = 5$

\# Special cases in graphical solution
1) Multiple optimal solution
2) No fesible solution
   1) Unbounded solution
   2) No feasible region

1) maximize $z = x_1 + 2x_2$

s.t $x_1 \leq 80$
$x_2 \leq 60$
$5x_1 + 6x_2 \leq 600$
$x_1 + 2x_2 \leq 160$
$x_1, x_2 \geq 0$

$\rightarrow 5x_1 + 6x_2 = 600$

Put $x_1 = 0$

$6x_2 = 600 \Rightarrow x_2 = 100$ (0, 100)

$x_2 = 0$

$5x_1 = 600 \Rightarrow x_1 = 120$ (120, 0)

$x_1 + 2x_2 = 160$

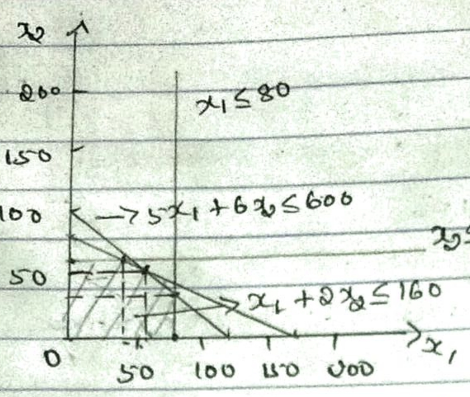
Put $x_1 = 0$

$2x_2 = 160 \Rightarrow x_2 = 80$ (0, 80)

$x_2 = 0$

$x_1 = 160$ (160, 0)

$x_1 = 80, \quad x_2 = 60$



$(0,0) \Rightarrow z = 0$

$(0,60) \Rightarrow z = 120$

$(40,60) \Rightarrow z = 160$

$(60,50) \Rightarrow z = 150$

$(80,30) \Rightarrow z = 140$

$(80,0) \Rightarrow z = 80$

$z = 160 \quad x_1 = 40 \quad x_2 = 60$

2) Maximize $z = 3x_1 + 2x_2$

$$x_1 - x_2 \le 1$$
$$x_1 + x_2 \ge 3$$
$$x_1, x_2 \ge 0$$

$\rightarrow \quad x_1 - x_2 \le 1$

$x_1 - x_2 = 1$

Put $x_1 = 0 \Rightarrow x_2 = -1$

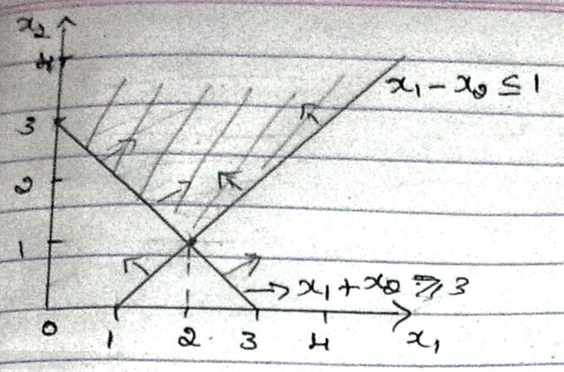
Put $x_2 = 0 \Rightarrow x_1 = 1$ $\quad (1,0)$

Put $x_1 = 2 \Rightarrow x_2 = 1$ $\quad (2,1)$

$x_1 + x_2 = 3$

Put $x_1 = 0 \Rightarrow x_2 = 3$ $\quad (0,3)$

$x_2 = 0 \Rightarrow x_1 = 3$ $\quad (3,0)$



Unbound solution. The feasible region is unbounded.
Can't find maximum point.

3) max $z = 200x_1 + 300x_2$

s.t. $2x_1 + 3x_2 \ge 1200$

$x_1 + x_2 \le 400$

$2x_1 + 1.5x_2 \ge 900$

$x_1, x_2 \ge 0$

$\rightarrow$ $2x_1 + 3x_2 = 1200$

Put $x_1 = 0$

$3x_2 = 1200 \Rightarrow x_2 = 400$ (0, 400)

$x_2 = 0$

$2x_1 = 1200 \Rightarrow x_1 = 600$ (600, 0)

$x_1 + x_2 \le 400$

Put $x_1 = 0 \Rightarrow x_2 = 400$ (0, 400)

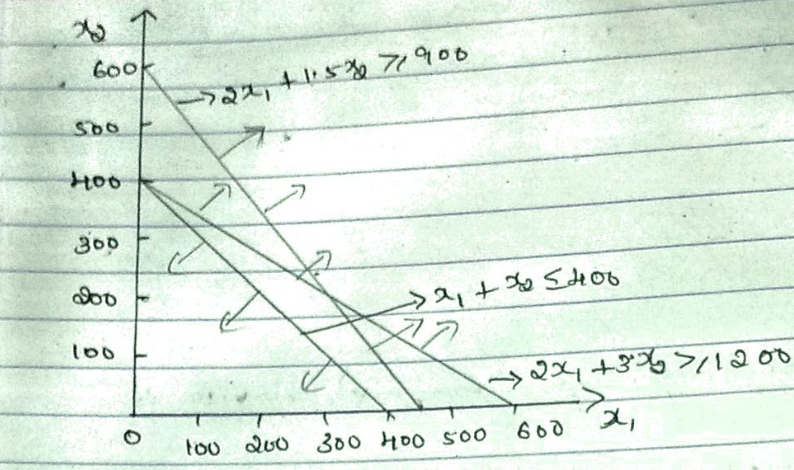
$x_2 = 0 \Rightarrow x_1 = 400$ (400, 0)

$2x_1 + 1.5x_2 \ge 900$

Put $x_1 = 0$

$1.5x_2 = 900 \Rightarrow x_2 = 600$ (0, 600)

$x_2 = 0 \Rightarrow 2x_1 = 900 \Rightarrow x_1 = 450$ (450, 0)



There is no common region which satisfies all the constraints. Therefore There is no feasible region for the graph. Hence the problem doesnot have any solution.

4) minimize $z = 1.5x_1 + 0.5x_2$

s.t $x_1 + 3x_2 \geq 3$

$x_1 + x_2 \geq 2$

$x_1, x_2 \geq 0$

$\to$ $x_1 + 3x_2 = 3$

Put $x_1 = 0$

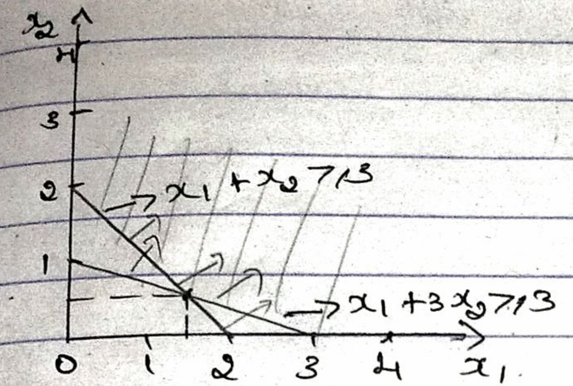
$3x_2 = 3 \Rightarrow x_2 = 1$ $(0,1)$

$x_2 = 0 \Rightarrow x_1 = 3$ $(3,0)$

$x_1 + x_2 \geq 2$

Put $x_1 = 0 \Rightarrow x_2 = 2$ $(0,2)$

$x_2 = 0 \Rightarrow x_1 = 2$ $(2,0)$



~~(0.5,~~ (1.5, 0.5) $\Rightarrow Z = 3.5$

Here we are finding minimum so least point needs to be considered.

Feasible region: Region which satisfies all the constraints

Optimal Solution

It is most favourable solution (highest value for maximization problems and least value for minimization problem.